Introduction to Mathematical Quantum Theory

Solution to the Exercises

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Exercise 1

Let V be a Banach space and E a nonempty subset of V such that for any $\xi \in V^*$ there exists a finite constant C_ξ such that

$$\sup_{x \in E} |\xi(x)| \leqslant C_\xi. \tag{1}$$

Prove that E must be bounded.

Hint: Consider the map $J : V \to V^{**}$ defined as

$$[J(x)](\xi) := \xi(x) \qquad \forall x \in V,\ \xi \in V^*. \tag{2}$$

Prove that $\|J(x)\|_{V^{**}} = \|x\|$ for any $x \in V$. Use the Uniform Boundedness Principle to show that $J(E)$ is bounded and conclude.

Proof. Consider $x \in V$. Recall that we proved that for any x we have

$$\|x\| = \sup_{\xi \in V^*,\ \|\xi\|_{V^*}=1} |\xi(x)|.$$

We then get

$$\|J(x)\|_{V^{**}} = \sup_{\xi \in V^*,\ \|\xi\|_{V^*}=1} |[J(x)](\xi)| = \sup_{\xi \in V^*,\ \|\xi\|_{V^*}=1} |\xi(x)| = \|x\|.$$

Consider now the set $J(E) \subseteq V^{**}$. Consider $\xi \in V^*$; using the hypothesis we then get

$$\sup_{x \in E} |[J(x)](\xi)| = \sup_{x \in E} |\xi(x)| \leqslant C_\xi.$$

We can then apply the uniform boundedness principle to get that there exists a constant C such that

$$\sup_{x \in E} \|J(x)\|_{V^{**}} \leqslant C.$$

As a consequence we get

$$\sup_{x \in E} \|x\| = \sup_{x \in E} \|J(x)\|_{V^{**}} \leqslant C,$$

and therefore E is bounded.

□

Exercise 2

Consider (X, Ω) a measurable space (i.e., a set X with a σ-algebra Ω in it), and consider a projection-valued measure with values in $\mathcal{H}$ an Hilbert space. Let $E,\ F \in \Omega$.

a Prove that if $E \cap F = \varnothing$ then $\operatorname{Ran} \mu(E) \perp \operatorname{Ran} \mu(F)$.

b Prove that $\mu(E)\mu(F)$ is an orthogonal projector and that

$$\operatorname{Ran}(\mu(E)\mu(F)) = \operatorname{Ran}\mu(E) \cap \operatorname{Ran}\mu(F). \tag{3}$$

Proof. To prove **a** first recall that from the definition of projection-valued measure we get that for any $E,\ F \in \Omega$ we have $\mu(E \cap F) = \mu(E)\mu(F)$. Therefore if $E \cap F = \varnothing$ we have that $\mu(E)\mu(F) = \mu(F)\mu(E) = \mu(\varnothing) = 0$. Let now $\psi \in \operatorname{Ran}\mu(E)$, $\phi \in \operatorname{Ran}\mu(F)$. Given that $\mu(E)$ and $\mu(F)$ are orthogonal projectors, we get $\psi = \mu(E)\psi$ and $\phi\mu(F)\phi$, and as a consequence

$$\langle \phi, \psi \rangle = \langle \mu(F)\phi, \mu(E)\psi \rangle = \langle \phi, \mu(F)^* \mu(E)\psi \rangle = \langle \phi, \mu(F)\mu(E)\psi \rangle = 0,$$

and therefore $\operatorname{Ran}\mu(E) \perp \operatorname{Ran}\mu(F)$.

To prove **b**, first we get that in general for any $E,\ F \in \Omega$ we get $\mu(E)\mu(F) = \mu(E \cap F)$, and given that the latter is an orthogonal projector, also the former is. To prove (3), we first prove $\subseteq$. Indeed we get trivially that $\operatorname{Ran}(\mu(E)\mu(F)) \subseteq \operatorname{Ran}\mu(E)$, and on the other hand $\operatorname{Ran}(\mu(E)\mu(F)) = \operatorname{Ran}(\mu(F)\mu(E)) \subseteq \operatorname{Ran}\mu(F)$, therefore it must be included in the intersection.

On the other hand, to prove $\supseteq$ let $\psi \in \operatorname{Ran}\mu(E) \cap \operatorname{Ran}\mu(F)$. Then we get that $\mu(E)\psi = \psi = \mu(F)\psi$. As a consequence we get $\psi = \mu(E)\psi = \mu(E)\mu(F)\psi \in \operatorname{Ran}(\mu(E)\mu(F))$, and this concludes the proof.

□

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Let A be a self-adjoint bounded operator over $\mathcal{H}$. Let B a bounded operator over $\mathcal{H}$ such that $[A, B] = 0$. Consider a bounded complex-valued measurable function f. Prove that $[f(A), B] = 0$.

Proof. Notice first that if $[A, B] = 0$ then $[A^n, B] = 0$ for any $n \in \mathbb{N}$. As a consequence, if f is a polynomial we also get $[f(A), B] = 0$. Consider now f a real-valued continuous function; from Weierstrass theorem we get that there exists a sequence of polynomials p_n that converges uniformly to f as n goes to infinity, and applying the result to the sequence of polynomials we get that also $f(A)$ commutes with B. Now, any complex-valued function f can be written as $f = \operatorname{Re} f + i \operatorname{Im} f$, and given that $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous and real-valued the result is also true for complex-valued continuous functions. Consider now the set $\mathcal{F} : \{f : \sigma(A) \to \mathbb{C} |\ [f(A), B]\}$; so far we proved that any complex-valued continuous function is in $\mathcal{F}$. Given that $\mathcal{F}$ is closed by uniformly bounded pointwise limit, we get that $\mathcal{F} = L^\infty(\sigma(A); \mathbb{C})$, which concludes the result.

□

Exercise 4

Let $\mathcal{H}$ be an Hilbert space. Let T be a bounded operator over $\mathcal{H}$. We proved in class that in general $R(T) \leqslant \|T\|$, where

$$R(T) := \sup_{\lambda \in \sigma(T)} |\lambda| . \tag{4}$$

Exhibit an explicit operator such that $R(T) < \|T\|$.

Proof. Consider the operator T defined on the Hilbert space $\mathcal{H} := L^2(I)$, with $I = (0,1)$ as

$$T\psi(x) := \int_0^1 \psi(x)\, dx.$$

T is a well-defined bounded linear operator and we proved in one of the exercise sessions that the spectrum of T is $\sigma(T) = \{0\}$, and therefore $R(T) = 0$. On the other hand, $T \neq 0$ implies $\|T\| > 0 = R(T)$.

□